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Relation Between the Kähler Equation and the Dirac Equation

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The formal analogy and the substantial differences between the Kähler equation and the Dirac equation are explained in terms of the relativistic compatibility of a common differential operator on the Clifford algebra C with two distinct representations of the Lorentz Lie algebra on C.

1. INTRODUCTION

Any first-order linear homogeneous relativistic field equation in Minkowski space-time is characterized by a vector space in which the fields take their values, a representation of the Lie algebra associated with the homogeneous Lorentz group specifying the transformation properties of the fields, and a differential operator annihilating the physically admissible fields. The relativistic invariance of the equation is guaranteed by certain relations between the operators of the representation and the coefficients of the differential operator.

The formal analogy between the Kähler equation and the Dirac equation, which has drawn some attention in the last two decades (Graf, 1978; Becher and Joos, 1982; Benn and Tucker, 1983; Talebaoui, 1994, 1995), can be transparently explained in this perspective, together with the substantial differences between the two equations.

We point out in this paper that the two linear representations of the Lorentz Lie algebra carried naturally by the Clifford algebra associated with the Minkowski metric specify two *distinct relativistic equations* associated with *the same differential operator.* One of the equations is just the Kähler equation, which decomposes uniquely into four inequivalent components of

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Duffin–Kemmer type. The other equation, which we call the *Clifford equation*, decomposes (in many ways) into four components, all equivalent to the Dirac equation. This accounts for the analogy and for part of the differences.

Further differences are made manifest by the remark that, unlike the decomposition of the Kähler equation, which is unique and carries through unaltered from Minkowski to curved space-time (so that a general Kähler field is just a set of four noninteracting meson fields), the decomposition of the Clifford equation does not hold in general in curved space-time, so that a Clifford field is essentially distinct from a set of four noninteracting Dirac fields.

2. EXTERIOR ALGEBRA, CLIFFORD ALGEBRA, AND THE IDENTIFICATION OF THEIR UNDERLYING LINEAR SPACES

Let M be the real four-dimensional Minkowski vector space, with metric tensor represented, with respect to any orthonormal basis, by the matrix $(\eta_{ih}) \equiv \mathrm{diag}(-1, -1, -1, 1)$ or, in contravariant form, by the matrix (η^{ih}) with the same entries. The indices run from 1 to 4.

The *exterior algebra* associated with M can be defined as the associative algebra Λ with unit determined by four generators a^1, a^2, a^3, a^4 with the relations

$$a^i \wedge a^h + a^h \wedge a^i = 0 \qquad (i, h = 1, 2, 3, 4) \tag{1}$$

where $\wedge$ denotes the product, called *exterior product*. This definition involves the dimension of the linear space M, but not the metric.

A basis of Λ is constituted by the following set $\{a\}$ of 16 elements:

$$\begin{array}{c} 1 \\ a^1 \quad a^2 \quad a^3 \quad a^4 \\ a^{12} \quad a^{13} \quad a^{14} \quad a^{23} \quad a^{24} \quad a^{34} \\ a^{123} \quad a^{124} \quad a^{134} \quad a^{234} \\ a^{1234} \end{array} \tag{2}$$

where

$$a^{ih} \equiv a^i \wedge a^h, \qquad a^{ihk} \equiv a^i \wedge a^h \wedge a^k, \qquad a^{1234} \equiv a^1 \wedge a^2 \wedge a^3 \wedge a^4 \tag{3}$$

If the four abstract elements $\{a^i\}$ are interpreted as a set $\{e^i\}$ of linearly independent vectors of the space M^* dual to M, the abstract algebra Λ just defined is identified with the algebra of covariant antisymmetric tensors of M, with the product given by the antisymmetrized tensor product, i.e., with the quotient of the tensor algebra over M^* with respect to the two-sided ideal generated by the elements $e^i \otimes e^h + e^h \otimes e^i$, where $\otimes$ denotes the tensor product. With such an interpretation, the basis elements (2) will be denoted by

$$\begin{matrix} 1 \\ e^1 \quad e^2 \quad e^3 \quad e^4 \\ e^{12} \quad e^{13} \quad e^{14} \quad e^{23} \quad e^{24} \quad e^{34} \\ e^{123} \quad e^{124} \quad e^{134} \quad e^{234} \\ e^{1234} \end{matrix} \tag{4}$$

and the basis as a whole by $\{e\}$. The subspaces of $\Lambda^{(q)}$ of Λ generated by the basis elements with q indices ($q = 0, 1, 2, 3, 4$) are constituted of *homogeneous forms of degree q*.

Any other basis $\{f^i\}$ of M^*, with $f^i = l^i_h e^h$ and (l^i_h) any regular matrix, is a set of generators of Λ satisfying relations analogous to (1), and the elements of the set $\{f\} \equiv \{1, f^i, f^{ih}, f^{ihk}, f^{1234}; i < h < k\}$ of Λ constructed from this basis by analogy with (2) and (3) constitute a basis of Λ. The two bases $\{e\}$ and $\{f\}$ give the same gradation to Λ, and the components of the homogeneous forms of degree q transform like the strict components of covariant antisymmetric tensors. All this holds, in particular, if the two bases $\{e^i\}$ and $\{f^i\}$ are orthonormal with respect to the Minkowski metric, as we shall always assume from now on. Consequently the matrix (l^i_h) introduced above will be assumed to represent a Lorentz transformation.

Similarly, the *Clifford algebra C* associated with M can be defined as the associative algebra with unit determined by four generators b^1, b^2, b^3, b^4 with the relations

$$b^i \vee b^h + b^h \vee b^i = 2\eta^{ih} \qquad (i, h = 1, 2, 3, 4) \tag{5}$$

where $\vee$ denotes the product, called *Clifford product*, and the right-hand side must be interpreted as the unit of the algebra multiplied by the number $2\eta^{ih}$. In contrast to the definition of the exterior algebra, the definition of the Clifford algebra involves the metric as well as the dimension of M.

A basis of C is constituted by the set $\{b\}$ of 16 elements defined by (2) and (3) with the replacement of a with b and of the exterior product with the Clifford product:

$$\begin{matrix} 1 \\ b^1 \quad b^2 \quad b^3 \quad b^4 \\ b^{12} \quad b^{13} \quad b^{14} \quad b^{23} \quad b^{24} \quad b^{34} \\ b^{123} \quad b^{124} \quad b^{134} \quad b^{234} \\ b^{1234} \end{matrix} \tag{6}$$

$$b^{ih} \equiv b^i \vee b^h, \qquad b^{ihk} \equiv b^i \vee b^h \vee b^k, \qquad b^{1234} \equiv b^1 \vee b^2 \vee b^3 \vee b^4 \quad (7)$$

If the four abstract elements $\{b^i\}$ are interpreted as the elements of an *orthonormal* basis $\{e^i\}$ of the space M^* dual to M, the abstract algebra C just defined is identified with the quotient of the tensor algebra over M^* with respect to the two-sided ideal generated by the elements $e^i \otimes e^h + e^h \otimes e^i - 2\eta^{ih}$. As for the exterior algebra, with such an interpretation the basis elements (6) will be denoted by $\{1, e^i, e^{ih}, e^{ihk}, e^{1234}; i < h < k\}$, and the basis as a whole by $\{e\}$. Any other orthonormal basis of M^* with elements $f^i = l^i_h e^h$ (where the coefficients l^i_h represent a Lorentz transformation) is a set of generators of C satisfying relations analogous to (5), and the set $\{f\}$ related to the basis $\{f^i\}$ as the set $\{e\}$ is related to $\{e^i\}$ is also a basis of C.

The correspondence between the two bases $\{e\}$ and $\{f\}$ of the Clifford algebra and the correspondence between the two bases of the exterior algebra that we denoted by the same symbols $\{e\}$ and $\{f\}$ are identical, because their derivations do not involve the relations which distinguish the two algebras [namely, relations (1) and (5) for equal values of the indices]. Therefore C and Λ can be identified as *linear spaces* (not as algebras!), in a Lorentz-invariant way, simply by identifying the respective bases that we have already denoted by the same symbols.

In particular this identification entails $e^i \vee e^h \equiv e^i \wedge e^h$ for $i \neq h$, while $e^i \vee e^i = \eta^{ii}$, so that the two algebraic operations are connected by $e^i \vee e^h = e^i \wedge e^h + \eta^{ih}$, equivalent to the relation

$$\mathscr{a} \vee \mathscr{b} = \mathscr{a} \wedge \mathscr{b} + \eta(\mathscr{a}, \mathscr{b}) \tag{8}$$

for any elements $\mathscr{a}$ and $\mathscr{b}$ of degree 1 in $\Lambda \equiv C$. More generally, for any homogeneous 1-form $\mathscr{a}$ and any inhomogeneous form $\mathscr{B}$ one has

$$\mathscr{a} \vee \mathscr{B} = \mathscr{a} \wedge \mathscr{B} + \imath_{\mathscr{a}}(\mathscr{B}) \tag{9}$$

where $\imath_{\mathscr{a}}$ is the usual *interior product* operator characterized as the linear operator on Λ which vanishes on scalars and transforms any decomposable q-form $\mathscr{b}^1 \wedge \mathscr{b}^2 \wedge \cdots \wedge \mathscr{b}^q$ into the $(q-1)$-form

$$\imath_{\mathscr{a}}(\mathscr{b}^1 \wedge \mathscr{b}^2 \wedge \cdots \wedge \mathscr{b}^q)$$
$$\equiv \sum_{i=1}^{q} (-1)^{i+1} \eta(\mathscr{a}, \mathscr{b}^i) \mathscr{b}^1 \wedge \mathscr{b}^2 \wedge \cdots \wedge \hat{\mathscr{b}}^i \wedge \cdots \wedge \mathscr{b}^q$$

(where $\hat{\mathscr{b}}^i$ denotes a suppressed factor) (Crumeyrolle 1990, p. 35). Incidentally, relation (9) suggests an extension of the definition of the interior product which allows the 1-form $\mathscr{a}$ to be replaced by an arbitrary inhomogeneous form $\mathscr{A}$. In fact one can set

$$\imath_{\mathscr{A}}(\mathscr{B}) = \mathscr{A} \vee \mathscr{B} - \mathscr{A} \wedge \mathscr{B}$$

3. THE NATURAL REPRESENTATIONS OF $\mathfrak{so}(3, 1)$ CARRIED BY THE CLIFFORD ALGEBRA

In this section we denote by V the linear space underlying the Clifford algebra C, identified with the linear space underlying the exterior algebra Λ as above. The symbols Λ and C will be reserved to V endowed with the respective associative algebra structures. We shall also denote by $C_{\mathscr{L}}$ the Lie algebra associated with C (i.e., the linear space V with the Lie product defined as the Clifford commutator $[a, b] \equiv a \vee b - b \vee a$). We endow V with a basis $\{e\}$ constituted of the elements listed in (4), and we interpret the four elements $\{e^i\}$ of degree 1 as an orthonormal basis of M^*, as above.

Setting, for $a < b$,

$$\lambda^{ab} \equiv \tfrac{1}{2}e^{ab} \equiv -\lambda^{ba} \tag{10}$$

we find that the elements λ^{ab} satisfy the commutation relations

$$[\lambda^{ab}, \lambda^{cd}] = \eta^{bc}\lambda^{ad} - \eta^{ac}\lambda^{bd} - \eta^{bd}\lambda^{ac} + \eta^{ad}\lambda^{bc} \tag{11}$$

which characterize the structure of the Lie algebra $\mathfrak{so}(3, 1)$ of the homogeneous Lorentz group. Therefore the subspace $V^{(2)} \subset V$ of degree 2 is a Lie subalgebra of $C_{\mathscr{L}}$ isomorphic to $\mathfrak{so}(3, 1)$, and two distinct representations of $\mathfrak{so}(3, 1)$, "reg" and "ad," acting on the same carrier space V, appear naturally:

"reg" is defined as the restriction to the Lie subalgebra $V^{(2)} \subset C_{\mathscr{L}}$ of the representation of $C_{\mathscr{L}}$ determined by the regular representation of C. Thus its action on V is

$$\text{reg}(a): \quad v \to a \vee v \tag{12}$$

($a \in V^{(2)}$, $v \in V$).

"ad" is defined as the restriction of the adjoint representation of $C_{\mathscr{L}}$ to its Lie subalgebra $V^{(2)}$. Thus its action on the carrier space V is

$$\text{ad}(a): \quad v \to a \vee v - v \vee a \equiv [a, v] \tag{13}$$

($a \in V^{(2)}$, $v \in V$).

The representation "ad" corresponds to the natural action of the homogeneous Lorentz group on differential forms. Under its action the homogeneous subspaces $\Lambda^{(q)} \equiv V^{(q)}$ of V are invariant, but the left ideals of C are not.

On the other hand, the regular representation of the algebra C can be decomposed in many ways into four subrepresentations (see Appendix), all equivalent to the spin representation by the very definition of the latter (Crumeyrolle, 1990; Benn and Tucker, 1987). Therefore under the representation "reg" the left ideals corresponding to any such decomposition are invariant, while the homogeneous subspaces of V are not.

Thus the decomposition of an inhomogeneous exterior form determined by a decomposition of C into minimal left ideals is not a Lorentz-invariant process. Though the component forms arising from such a decomposition belong to subspaces that transform like Dirac spinors under the action of the representation "reg", they cannot be properly regarded as spinors because their own transformation law is given by the representation "ad". (The situation is reminiscent of the decomposition of a vector with respect to a basis, where the components cannot be properly regarded as scalars in the actual vector representation, though they belong to one-dimensional subspaces which would behave as scalars if the vector representation were replaced by the trivial representation of the group.)

4. THE RELATIVISTIC EQUATIONS ASSOCIATED WITH THE NATURAL REPRESENTATIONS

The above remarks on the lack of a relativistically significant algebraic relationship between spinors and exterior forms apply, of course, to *spinor fields* and *differential forms* as well. There is, however, a well-known affinity between the Dirac equation and the Kähler equation, whose origin will be exhibited within the general classification scheme for first-order linear homogeneous relativistically invariant field equations.

Let $\Phi \equiv \Phi(x)$ denote a field defined in Minkowski space-time, with values in the representation space S of some linear representation π of $Sl(2, C)$ (x stands for the Cartesian coordinates x^1, x^2, x^3, x^4 associated with a reference frame with basis vectors e^1, e^2, e^3, e^4). Recall (Gel'fand *et al.*, 1963) that the Lorentz invariance of an equation of the form

$$L^h \frac{\partial \Phi}{\partial x^h} + k\Phi = 0 \tag{14}$$

(where L^1, L^2, L^3, L^4 are four linear operators on S, and k is a constant) is guaranteed by the commutation relations

$$[L^h, \pi^{ab}] = \eta^{ah} L^b - \eta^{bh} L^a \tag{15}$$

The Lie algebra $\mathfrak{so}(3, 1)$ is identified as usual with $\mathfrak{sl}(2, C)$, and in (15) the operators $\pi^{ab} \equiv \pi(l^{ab})$ denote the representatives of a basis (l^{ab}) of $\mathfrak{so}(3, 1)$ with respect to which the structural relations of the Lie algebra have the same form as in (11), namely

$$[l^{ab}, l^{cd}] = \eta^{bc} l^{ad} - \eta^{ac} l^{bd} - \eta^{bd} l^{ac} + \eta^{ad} l^{bc}$$

If a representation π is given, to construct a relativistic equation of type (14) it is sufficient to determine four operators L^h acting on S and satisfying relations (15) with the operators π^{ab}.

Now consider the vector space V underlying the Clifford algebra C, endowed with a basis $\{e\}$, and the two representations "reg" and "ad" of the

Lorentz Lie algebra defined in Section 3. It is easy to check that the same set of four operators L^h defined by

$$L^h\colon a \rightarrow e^h \vee a \tag{16}$$

satisfies relations (15) with the operators π^{ab} of both representations. In fact, if π is identified with the representation "reg", one has, from the definitions (10) and (12) and using the anticommutation relations (5),

$$\begin{aligned}[e^h, \lambda^{ab}] &= [e^h, \tfrac{1}{2}e^a \vee e^b] \\ &= \tfrac{1}{2}(e^h \vee e^a \vee e^b - \tfrac{1}{2}e^a \vee e^b \vee e^h \\ &= \eta^{ah}e^b - \eta^{bh}e^a\end{aligned}$$

Similarly, if π is identified with "ad", one has, for $v \in V$,

$$\begin{aligned}&e^h \vee [\lambda^{ab}, v] - [\lambda^{ab}, e^h \vee v] \\ &\quad = \tfrac{1}{2}(e^h \vee e^a \vee e^b \vee v - e^a \vee e^b \vee e^h \vee v) \\ &\quad = (\eta^{ah}e^b - \eta^{bh}e^a) \vee v\end{aligned}$$

Thus in both cases the relations (15) hold.

Consequently, when Φ takes its values in the Clifford algebra, the equation

$$e^h \vee \frac{\partial \Phi}{\partial x^h} + k\Phi = 0 \tag{17}$$

which cannot be regarded as a relativistic equation until the action of $\mathfrak{so}(3, 1)$ on C has been specified, turns out to describe two *distinct* relativistic equations according to whether the representation "ad" or the representation "reg" is adopted.

The first choice is just the Kähler equation, because, on account of (9), the differential operator $e^h \vee \partial/\partial x^h$ turns out to be identical with the Kähler operator $d - \delta$, where d and δ denote, respectively, the operators of differentiation and codifferentiation of differential forms. It turns out that the Kähler equation decomposes uniquely into four equations of Duffin–Kemmer type (Cantoni, 1996).

The second choice gives rise to the relativistic equation that we shall call the *Clifford equation*. If one chooses a basis of C adapted to one of its decompositions into minimal left ideals (see Appendix), the Clifford equation decomposes into four components equivalent to the Dirac equation (Talebaoui, 1995).

5. THE "REGULAR" CLIFFORD BUNDLE AND THE "CLIFFORD EQUATION"

The above remarks can be summarized as follows: we are in the presence of *two distinct* fiber bundles, both with Minkowski space-time as base space, the Clifford algebra C as typical fiber, the homogeneous Lorentz group as structure group. They differ by the actions of the group on the fiber, which are distinct.

One of the actions, which might be called *bosonic*, defines a *Clifford bundle* (Crumeyrolle, 1990) and corresponds to the Kähler equation, whose decomposition gives rise to the Duffin–Kemmer equations, so that a Kähler field is just a set of four independent fields: scalar, pseudoscalar, vector, and pseudovector (Cantoni, 1996).

The other action, which might be called *fermionic*, defines what we shall call a *regular Clifford bundle* (on account of its relation with the regular representation), and corresponds to what we have already called the *Clifford equation*, whose decomposition in Minkowski space gives rise to four copies of the Dirac equation.

The two equations further differ significantly with regard to the extendibility of their decomposition properties: in the unique decomposition of the Kähler equation into four equations of Duffin–Kemmer type the flatness of the base space is irrelevant, whereas the decomposition of the covariant version of the Clifford equation, which is possible in many ways on Minkowski space, becomes impossible, in general, if the base space is endowed with curvature. Therefore the Clifford field is not a mere juxtaposition of four independent Dirac fields.

Finally we notice that, on a curved base space, the very identity of the differential operators which was at the origin of the analogies between the Kähler equation and the Dirac equation is lost, on account of the undisguisable difference in the connection coefficients, which are essentially dependent on the transformation properties of the fields.

APPENDIX. THE DECOMPOSITIONS OF THE REGULAR REPRESENTATION

Take any set $\{\gamma^h\}$ of Dirac matrices and use them to represent the Clifford algebra C by the algebra $\mathscr{M}$ of 4×4 matrices via the following correspondence from the basis elements (6) to the matrices: $1 \to I$ (unit matrix), $b^h \to \gamma^h$, $b^{hk} \to \gamma^h\gamma^k$, $b^{hkl} \to \gamma^h\gamma^k\gamma^l$, $b^{hklm} \to \gamma^h\gamma^k\gamma^l\gamma^m$. Since the matrices corresponding to the basis elements of C are a basis of $\mathscr{M}$, the four matrices $\underset{r}{E}$ ($r = 1, 2, 3, 4$), defined as the matrices with all the entries equal to zero except for the intersection of the rth row with the rth column, which is equal to 1, represent four primitive idempotents $e(r)$ of C. The corresponding minimal left ideals are represented by the four sets $\{m\underset{r}{E};\ m \in \mathscr{M}\}$ of matrices with nonzero entries only in the rth column, and in the corresponding decomposition of $\mathscr{M}$ the generic 4×4 matrix m is expressed as the sum of four matrices with three columns of zeros and the remaining column equal to the corresponding one in m.

For any regular 4×4 matrix T, the four matrices $T\underset{r}{E}T^{-1}$ represent an alternative choice of pairwise orthogonal idempotents. Distinct choices of T lead, in general, to distinct decompositions of C.

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